

# The dyon charge in noncommutative gauge theories

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## Abstract

We present an explicit classical dyon solution for the noncommutative version of the Yang-Mills-Higgs model (in the Prasad-Sommerfield limit) with a  $\vartheta$ -term. We show that the relation between classical electric and magnetic charges also holds in noncommutative space. Extending the Noether approach to the case of a noncommutative gauge theory, we analyze the effect of CP violation at the quantum level, induced both by the  $\vartheta$  term and by noncommutativity and we prove that the Witten effect formula for the dyon charge remains the same as in ordinary space.

## 1 Introduction

Gauge theories coupled to Higgs scalars exhibit a remarkable phenomenon, usually called Witten effect [1], related to the  $\vartheta$ -angle. Indeed, if one adds

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to a Yang-Mills-Higgs Lagrangian, a  $\vartheta$ -term,

$$\Delta L = \vartheta \frac{e^2}{32\pi^2} \varepsilon^{\mu\nu\alpha\beta} \text{tr} (F_{\mu\nu} F_{\alpha\beta}) , \quad (1)$$

which explicitly violates CP, the electric charge  $q_e$  of a dyon is modified. Instead of being quantized, as in the  $\vartheta = 0$  case, in units of the fundamental charge  $e$  - as can be seen using, for example, semiclassical quantization arguments [2]- one has, for  $\vartheta \neq 0$

$$q_e = \left( n + \frac{\vartheta}{2\pi} \right) e , \quad n \in Z \quad (2)$$

This result corresponds to a Julia-Zee dyon [3] with magnetic charge  $m = 4\pi/e$ . There are also arguments leading to the conjecture that other CP violating interactions may also induce a shift of the dyon charge [1].

CP violation can be induced not just by adding new interactions to the Yang-Mills-Higgs Lagrangian but by radically changing the setting of the theory. This is the case of noncommutative gauge theories (NCGT) where the introduction of noncommutation in space-time coordinates has shown to affect the behavior under C, P and T invariance [4]-[7]. More specifically, one can prove that when noncommutativity is restricted to space coordinates,

$$\begin{aligned} [x_i, x_j] &= i\theta_{ij} , \quad i, j = 1, 2, 3 \\ [x_i, x_0] &= 0 \end{aligned} \quad (3)$$

NCGT are not charge invariant. Only if the usual field transformations are accompanied by a change of sign in  $\theta_{ij}$ , charge invariance is recovered. Hence, if one takes  $\theta_{ij}$  as a fixed parameter which does not transform as fields do, CP is violated, although CPT invariance is maintained since parity invariance is not affected by the introduction of  $\theta$  and time reversal undergoes a change that compensates that in C.

It is then natural to pose the question whether the dyon charge in noncommutative gauge theories receives a contribution from a CP violating effect induced by noncommutativity even if the  $\vartheta$  angle vanishes. Moreover, one

could also ask how the addition of the noncommutative version of the term (1) modifies  $Q$  when both  $\theta_{ij} \neq 0$  and  $\vartheta \neq 0$ .

We analyze these questions in the present paper and, to this end, we first discuss the properties of the dyon in a noncommutative Yang-Mills-Higgs model with  $U(2)$  gauge symmetry, calculate its charge at the quantum level and also extend the theory in order to include a  $\vartheta$  term. Some of these issues were briefly discussed in [8] starting from a monopole solution obtained generalizing Nahm's equations that describes BPS solitons [9] (Some aspects of dyon solutions were also considered in ref.[10]). Here, instead, we shall extend the more explicit  $U(2)$  monopole solution found in [12]-[13] to the case of a dyon and then establish a noncommutative version of the Noether theorem in order to define the operator  $\mathcal{N}$  which generates the  $U(1)$  gauge transformations associated with electric charge. We then discuss the issue of the Witten effect in noncommutative space.

## 2 The Bogomolnyi bound and the noncommutative dyon equations

The action for the noncommutative  $U(2)$  Yang-Mills-Higgs system that we consider is

$$S = \text{tr} \int d^4x \left( -\frac{1}{2} F_{\mu\nu} * F^{\mu\nu} + D_\mu \Phi * D^\mu \Phi \right) \quad (4)$$

Gauge fields  $A_\mu = A_\mu^A t^A$  take values in the Lie algebra of  $U(2)$  with generators  $t^A$ ,  $A = 0, 1, 2, 3$  ( $t^0 = I/2$ ,  $t^a = \sigma^a/2$ ,  $a = 1, 2, 3$ ).  $\Phi = \Phi^A t^A$  is the Higgs multiplet and we consider the Prasad-Sommerfield limit [11] in which the symmetry breaking potential vanishes. Covariant derivatives and field strength are defined as follows

$$\begin{aligned} D_\mu \Phi &= \partial_\mu \Phi - ie[A_\mu, \Phi]_* \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - ie[A_\mu, A_\nu]_* \end{aligned} \quad (5)$$

The star product  $*$  in (4) is defined as usual

$$A(x) * B(x) = \exp \left( \frac{i}{2} \theta_{\mu\nu} \partial_x^\mu \partial_y^\nu \right) A(x) B(y) \Big|_{y=x} \quad (6)$$

and provides a simple way to handle field theories in noncommutative space as that defined in (3). We have defined

$$[A(x), B(x)]_* = A(x) * B(x) - B(x) * A(x) \quad (7)$$

As in eq.(3) we shall take  $\theta_{0i} = 0$ . This ensures a well-defined Hamiltonian and unitarity at the quantum level. Moreover, under such conditions, we shall see that the Noether theorem can be naturally extended to the noncommutative case and conserved charges can be derived avoiding the problems implied by the infinite number of time derivatives that introduces  $\theta_{0i} \neq 0$ .

Let us briefly recall at this point the way in which the  $*$ -product induces charge violation. This can be easily seen just by analyzing the pure gauge action. Under charge conjugation gauge fields change according to

$$A_\mu = A_\mu^a t^a \xrightarrow{C} A_\mu^C = -\overline{A}_\mu = -A_\mu^a \overline{t}^a \quad (8)$$

where  $\overline{A}_\mu$  represents the complex conjugate representation. One can easily see [5] that already the commutator  $[A_\mu^0, A_\nu^0]$  entering in the  $F_{\mu\nu}^0$  component of the field strength does not change sign under charge conjugation while the corresponding derivative terms do. There are also mismatches concerning the  $SU(2)$  components. Only if change (8) is accompanied by a change of sign in  $\theta_{ij}$  these changes are compensated. Since we take  $\theta_{ij}$  as fixed parameters, we have

$$\text{tr}(F_{\mu\nu} F_{\mu\nu}) \neq \text{tr}(F_{\mu\nu}^C F_{\mu\nu}^C) \quad (9)$$

and then the action (4) is not invariant under charge conjugation. In contrast, since under parity transformation one has

$$A_\mu \xrightarrow{P} A_\mu^P = \begin{cases} A_0 \\ -A_i \end{cases} \quad (10)$$

it is easy to see that these transformations together with those in coordinates  $(x^i \xrightarrow{P} -x^i)$  leave the action (4) invariant. Regarding time inversions, one can see that the action changes and such change, compensates the one produced by charge conjugation. In summary, CP is violated but the theory is CPT invariant.

We shall now look for noncommutative dyon solutions which, being static, can be found by searching the minima of the energy, defined as

$$E = \text{tr} \int d^3x (E_i * E_i + B_i * B_i + D_i \Phi * D_i \Phi + D_0 \Phi * D_0 \Phi) \quad (11)$$

where we have written

$$E_i = -F_{0i} \ , \quad B_i = -\frac{1}{2} \varepsilon_{ijk} F^{jk} \quad (12)$$

Note that since we are working in the BPS limit, the vacuum expectation value is no longer determined by the Lagrangian but imposed as a boundary condition on the Higgs field

$$\text{tr} \Phi_{vac}^2 = \frac{v_0^2}{2} \quad (13)$$

As in ordinary space[14], Eq.(11) can be written in the form

$$\begin{aligned} E = & \text{tr} \int d^3x ((E_i - \sin \alpha D_i \Phi) * (E_i - \sin \alpha D_i \Phi) \\ & + (B_i - \cos \alpha D_i \Phi) * (B_i - \cos \alpha D_i \Phi) + D_0 \Phi * D_0 \Phi \\ & + 2 \sin \alpha E_i * D_i \Phi + 2 \cos \alpha B_i * D_i \Phi) \end{aligned} \quad (14)$$

Thus, one has a Bogomol'nyi bound on the energy

$$E \geq v_0 \sin \alpha Q + v_0 \cos \alpha M \quad (15)$$

with  $Q$  and  $M$  the electric and magnetic charges defined as

$$\begin{aligned} Q &= \frac{2}{v_0} \text{tr} \int d^3x E_i * D_i \Phi \\ M &= \frac{2}{v_0} \text{tr} \int d^3x B_i * D_i \Phi \end{aligned} \quad (16)$$

The bound is saturated whenever the following BPS equations hold

$$E_i = \sin \alpha D_i \Phi \quad (17)$$

$$B_i = \cos \alpha D_i \Phi \quad (18)$$

$$D_0 \Phi = 0 \quad (19)$$

Since we are looking for static configurations, eq.(19) implies

$$[A_0, \Phi]_* = 0 \quad (20)$$

In order to find an explicit dyon solution we consider below an expansion of fields  $A_\mu$  and  $\Phi$  in powers of the noncommutative parameter in  $\theta$  thus extending to the dyon case the approach developed in [12] and [13] where a purely magnetically charged solution was found, to first and second order in  $\theta$  respectively.

### 3 Dyon solution as a $\theta$ -expansion

The approach in refs.[12]-[13] for obtaining a non-commutative monopole solution starts from the exact Prasad-Sommerfield solution in ordinary space [11] as giving the zeroth-order of an expansion in powers of  $\theta$  for the monopole in noncommutative space. Plugging this expansion into the BPS equations, one obtains the noncommutative solution order by order in  $\theta$ . We shall follow this approach but including the  $A_0$  component of the gauge field so as to solve eqs. (17)-(19) and construct the dyon solution.

We then take as zeroth order approximation for the  $SU(2)$  components the Prasad-Sommerfield dyon solution [11],

$$\begin{aligned} \Phi^{a(0)} &= \left( v_0 \coth(v_0 e \cos(\alpha) r) - \frac{1}{er \cos(\alpha)} \right) \frac{x^a}{r} \\ A_i^{a(0)} &= \cos(\alpha) \left( \frac{1}{er \cos(\alpha)} - \frac{v_0}{\sinh(v_0 er \cos(\alpha))} \right) \epsilon_{aij} \frac{x^j}{r} \\ A_0^{a(0)} &= \sin(\alpha) \left( v_0 \coth(v_0 e \cos(\alpha) r) - \frac{1}{er \cos(\alpha)} \right) \frac{x^a}{r} \end{aligned} \quad (21)$$

Notice that

$$A_0^{a(0)} = \sin \alpha \Phi^{a(0)} \quad (22)$$

Concerning the  $U(1)$  components, we take

$$\Phi^{0(0)} = 0, \quad A_i^{0(0)} = 0, \quad A_0^{0(0)} = 0, \quad (23)$$

In order to find the complete solution we write

$$\begin{aligned} \Phi &= (\Phi^{a(0)} + \tilde{\Phi}^{a(1)} + \tilde{\Phi}^{a(2)})t_a \\ &+ (\Phi^{0(0)} + \tilde{\Phi}^{0(1)} + \tilde{\Phi}^{0(2)})t_0 + \mathcal{O}(\theta^3) \\ A_i &= (A_i^{a(0)} + \tilde{A}_i^{a(1)} + \tilde{A}_i^{a(2)})t_a \\ &+ (A_i^{0(0)} + \tilde{A}_i^{0(1)} + \tilde{A}_i^{0(2)})t_0 + \mathcal{O}(\theta^3) \\ A_0 &= (A_0^{a(0)} + \tilde{A}_0^{a(1)} + \tilde{A}_0^{a(2)})t_a \\ &+ (A_0^{0(0)} + \tilde{A}_0^{0(1)} + \tilde{A}_0^{0(2)})t_0 + \mathcal{O}(\theta^3) \end{aligned} \quad (24)$$

To first order in  $\theta$ , an ansatz for the gauge potencial and Higgs field components on  $U(1)$ , that obeys covariance under the  $SO(3)$  rotation corresponding to the diagonal subgroup of  $SO(3)_{gauge} \times SO(3)_{space}$  is

$$\begin{aligned} \tilde{A}_i^{0(1)} &= \theta_{ij}x_j A(r) + \varepsilon_{ijk}\theta_{jk}C(r) + x_i\varepsilon_{jkl}\theta_{jk}x_l D(r) \\ \tilde{A}_0^{0(1)} &= \theta_{ij}\varepsilon_{ijk}x_k K(r) \\ \tilde{\Phi}^{0(1)} &= \theta_{ij}\varepsilon_{ijk}x_k B(r) \end{aligned} \quad (25)$$

where  $A(r), B(r), C(r), D(r)$  and  $K(r)$  are radial functions to be determined. The component on  $SU(2)$  to first order in  $\theta$  of the Bogomol'nyi equation is not going to be analyze, due to it have a solution that is pure gauge.

Using ansatzæ (21) and (25) one can easily show that Bogomol'nyi eq.(20) implies, at this order in  $\theta$ , that

$$K(r) = \sin \alpha B(r) \quad (26)$$

Extending the symmetric ansatzæ(21)-(25) order by order in  $\theta$  one can prove that this kind of relation remains valid so that one can conclude that, to all orders in  $\theta$ , one has

$$A_0 = \sin \alpha \Phi \quad (27)$$

A similar relation was proposed in [8] as an ansatz within Nahm's approach to the construction of monopole solutions. Here, it was derived as a result of taking the original dyon solution in ordinary space as the zeroth order in a  $\theta$ -expansion.

Identity (27) implies that the BPS equation (17) is automatically satisfied and then the BPS system (17)-(18) reduces to

$$\frac{1}{2}\varepsilon_{ijk}F_{jk} = -\cos\alpha D_i\Phi \quad (28)$$

Except for the factor  $\cos\alpha$ , this is nothing but the pure noncommutative monopole equation. We can then use the pure monopole solutions constructed in [12]-[13] after making the appropriate rescaling. We find

$$\begin{aligned} \tilde{A}_i^{(n)}(x^i) &= \cos^{2n+1}\alpha A_i^{(n)}(\cos\alpha x^i), \quad n = 1, 2, \dots \\ \tilde{\Phi}(x^i)^{(n)} &= \cos^{2n}\alpha \Phi^{(n)}(\cos\alpha x^i), \quad n = 1, 2, \dots \end{aligned} \quad (29)$$

where  $A_i^{(n)}$  and  $\Phi^{(n)}$  are the pure monopole solutions found in [13]. Then, using eq.(27)  $\tilde{A}_0^{(n)}(x^i)$  can be constructed. We give their explicit form, to order  $\theta^2$ , in an Appendix.

An important property of the solution we found is that, having started at zeroth order with  $\Phi^{0(0)} = 0$ , one finds that this condition holds to all orders in  $\theta$ . It should be noticed that, as we already pointed, since we are working in the Prasad-Sommerfield limit, the vacuum expectation vacuum is no longer determined by the Lagrangian but imposed as a boundary condition on the Higgs field, eq.(13). The vanishing of the  $U(1)$  component of the Higgs field solution then implies that (13) should be guaranteed by the  $SU(2)$  components

$$(\Phi_{vac}^a)^2 = v_0^2, \quad \Phi_{vac}^0 = 0 \quad (30)$$

We are now ready to compute the electric and magnetic charges of the



dyon solution. Expanding the r.h.s of eqs.(16) we have

$$\begin{aligned}
Q &= \frac{2}{v_0} \text{tr} \int_{S_\infty^2} dS^i (E_i \Phi + \frac{i}{2} \theta^{lm} \partial_l E_i \partial_m \Phi - \frac{1}{4} \theta^{pq} \theta^{lm} \partial_p \partial_l E_i \partial_q \partial_m \Phi) + \mathcal{O}(\theta^3) \\
M &= -\frac{2}{v_0} \text{tr} \int_{S_\infty^2} dS^i (\frac{1}{2} \epsilon_{ijk} F_{jk} \Phi + \frac{i \epsilon_{ijk}}{4} \theta^{lm} \partial_l F_{jk} \partial_m \Phi \\
&\quad - \frac{\epsilon_{ijk}}{4} \theta^{pq} \theta^{lm} \partial_p \partial_l F_{jk} \partial_q \partial_m \Phi) + \mathcal{O}(\theta^3)
\end{aligned} \tag{31}$$

In computing these charges we shall need to know the asymptotic behavior of solutions (24), order by order in  $\theta$ . Starting from the behavior to order zero

$$\Phi^{a(0)} = v_0 + \mathcal{O}(e^{-v_0 r}) \tag{32}$$

$$A_i^{a(0)} = \frac{1}{er} + \mathcal{O}(1/r^2) + \mathcal{O}(e^{-v_0 r}) \tag{33}$$

$$A_0^{a(0)} = v_0 \sin \alpha + \mathcal{O}(e^{-v_0 r}) \tag{34}$$

$$\tag{35}$$

one has, to order  $n$  ( $n = 1, 2, \dots$ )

$$\Phi^{(n)} = \frac{f_0^{(n)}}{r^{n+1}} + \mathcal{O}(1/r^{n+2}) + \mathcal{O}(e^{-v_0 r}) \tag{36}$$

$$A_i^{(n)} = \frac{f_i^{(n)}}{r^{n+1}} + \mathcal{O}(1/r^{n+2}) + \mathcal{O}(e^{-v_0 r}) \tag{37}$$

$$A_0^{(n)} = \frac{f_0^{(n)} \sin \alpha}{r^{n+1}} + \mathcal{O}(1/r^{n+2}) + \mathcal{O}(e^{-v_0 r}) \tag{38}$$

with  $f_0^{(n)}$  and  $f_i^{(n)}$  constants. From this behavior one can see that  $n > 0$  orders in  $\theta$  do not contribute to the charges that thus coincide with those in ordinary space,

$$Q = \frac{1}{v_0} \int_{S_\infty^2} dS^i (\partial_i A_0^{a(0)} + e \epsilon_{abc} A_i^{b(0)} A_0^{c(0)}) \Phi^{a(0)} = \frac{4\pi}{e} \tan \alpha \tag{39}$$

$$M = -\frac{1}{v_0} \int_{S_\infty^2} dS^i \epsilon_{ijk} (\partial_j A_k^{a(0)} + \frac{e}{2} \epsilon_{abc} A_j^{b(0)} A_k^{c(0)}) \Phi^{a(0)} = \frac{4\pi}{e} \tag{40}$$

It could be argued that this coincidence is just a consequence of having constructed the solution as a power expansion in  $\theta$ , which has dimensions of  $[\text{length}]^2$ , so that higher orders in  $\theta$  would necessarily imply, asymptotically, higher powers in  $1/r$ . There is however another dimensional parameter,  $v_0^2$  ( $[v_0^2] = [1/\text{length}]^2$ ) so that (38) is the result of the structure of BPS equations and the boundary conditions and not just a dimensional question.

Note that the following relation between charges holds

$$Q = M \tan \alpha , \quad (41)$$

the same one satisfied by dyon electric and magnetic charges in ordinary space. We then conclude that CP violation induced by noncommutativity does not change, at least at the classical level, the charge of the dyon: no trace of  $\theta_{ij}$  appears in  $Q$ .

In order to analyze charge quantization at the quantum level, we can use the Noether approach and canonically proceed as originally done in [1] regarding the unbroken symmetry which leaves the Higgs vacuum invariant and is associated to the electric charge.

## 4 The Noether theorem and the dyon charge shift in the noncommutative case

In order to analyze the Noether charge we shall not just consider action (4) but one including the noncommutative version of a  $\vartheta$ -term,

$$S = \text{tr} \int d^4x \left( -\frac{1}{2} F_{\mu\nu} * F^{\mu\nu} + D_\mu \Phi * D^\mu \Phi + \vartheta \frac{e^2}{16\pi^2} F_{\mu\nu} * \tilde{F}^{\mu\nu} \right) \quad (42)$$

In this way, we shall be able to test possible modifications of the dyon charge both because noncommutativity and because the  $\vartheta$  term. (Of course the case of action (4) can be analyzed just by putting  $\vartheta = 0$ ). As in ordinary space, the noncommutative version of the  $\vartheta$  term can be written as a surface term [15] and hence the equations of motion for action (42) and its dyon solutions remain the same for all values of  $\vartheta$ .

After symmetry breaking through condition (30), the unbroken symmetry is related to rotations  $\Lambda$  in the direction of  $\Phi$ , which at infinity satisfy

$$\Lambda_\infty = \frac{1}{v_0} \Phi_{vac}^a t^a \quad (43)$$

clearly leaving the Higgs vacuum invariant. We thus consider such gauge transformations along the Higgs field direction

$$\Lambda(x) = \frac{1}{v_0} \Phi^a t^a \epsilon(x) \quad (44)$$

with  $\epsilon(\infty) = 1$ . With this, one has

$$\begin{aligned} \delta_\Lambda \Phi &= 0 \\ \delta_\Lambda A_\mu &= \frac{1}{ev_0} D_\mu \Phi \end{aligned} \quad (45)$$

When one looks for field transformations leaving the action  $S$  unchanged in order to find a conserved current in the noncommutative case, one has to take into account  $*$ -commutators that, once integrated, give a vanishing contribution, as first noticed for a scalar theory in [16]. To see this, let us consider a gauge transformation  $\Lambda$  with infinitesimal parameter  $\epsilon(x)$ . Being the action  $S$  invariant in the case  $\epsilon = \text{constant}$ , one can write for the local case

$$\delta_{\Lambda(x)} S = - \int d^4x J^\mu [\Phi(x), A_\mu(x)] \partial_\mu \epsilon(x) \quad (46)$$

or, after integrating by parts

$$\delta_{\Lambda(x)} S = \int d^4x \partial_\mu J^\mu [\Phi(x), A_\mu(x)] \epsilon(x) \quad (47)$$

Invariance under local transformations implies  $\delta_{\Lambda(x)} S = 0$  or

$$\int d^4x \partial_\mu J^\mu [\Phi(x), A_\mu(x)] \epsilon(x) = 0 \quad (48)$$

It is at this point that the Noether's procedure deviates from the usual one in ordinary space. Indeed, in the noncommutative case the most one can infer from eq.(48) is that

$$(\partial_\mu J^\mu) = \text{tr} ([O, P]_* + [B, C]_* * B + \dots) \quad (49)$$

for some proper functionals  $O, P, B, C, \dots$  since, once integrated over space-time, the r.h.s. in (49) vanishes due to the  $*$ -product cyclic properties under integration.

In order to find a conserved charge one has to integrate (49) over 3-space. Now, since we have taken  $\theta_{0i} = 0$  also in this case the integral of commutators vanishes,

$$\text{tr} \int d^3x ([O, P]_* + [B, C]_* * B + \dots) = 0 \quad (50)$$

so that, after integrating (49) one ends with a conserved charge of the form

$$N = \int d^3x J^0 \quad (51)$$

Use of the equations of motion allows to explicitly find

$$J^0 = -\frac{1}{ev_0} \text{tr} \left( 2F^{0i} * D_i \Phi - \frac{\vartheta e^2}{4\pi^2} \tilde{F}^{0i} * D_i \Phi \right) \quad (52)$$

This implying, in the present case

$$N = \frac{1}{ev_0} \text{tr} \int d^3x \left( 2F_{0i} * D_i \Phi - \frac{\vartheta e^2}{8\pi^2} \varepsilon_{ikj} F_{jk} * D_i \Phi \right) = -\frac{1}{e} Q + \frac{\vartheta e}{8\pi^2} M \quad (53)$$

At large distances, where  $\text{tr} \Phi^2 = v_0^2/2$  the operator  $\exp(2\pi i N)$  is a  $2\pi$  rotation about the direction of  $\Phi$  (elsewhere the rotation angle is  $2\pi|\Phi(x)|/v_0$  but by Gauss law if the gauge transformation is 1 at infinity it leaves the physical states invariant) . Then we have

$$\exp(2\pi i N) = 1 \quad (54)$$

and thus the eigenvalues of  $N$  have to be quantized in integer units  $n$ . If we call  $q_e$  and  $q_g$  the eigenvalues of the electric and magnetic charge operators, one gets from (53)

$$q_e = \left( ne + \frac{\vartheta e^2}{8\pi^2} q_g \right) \quad (55)$$

That is, we have obtained for the noncommutative dyon the same formula that holds for the case of ordinary space, eq.(2).

In summary, we have constructed an explicit noncommutative dyon solution showing that the relation (41) between classical electric and magnetic charge also holds in noncommutative space. Moreover, after extending the Noether approach to the case of a noncommutative gauge theory, we have proven that the  $\theta_{ij}$  dependent CP violation introduced by the commutation rule (3) does not change the Witten effect formula; indeed, the dyon's charge shift is  $\theta_{ij}$ -independent for constant parameters  $\theta_{ij}$ . In this respect, it should be interesting to consider other type of noncommutativity and in particular, to investigate the case of the dyon in the fuzzy sphere along the lines developed in ref.[17] where monopole solutions were constructed for the case in which  $\theta_{ij} = \theta r \varepsilon_{ijk} x_k$  since in that case the coordinate dependence of  $\theta_{ij}$  may introduce definite changes in (55). We hope to discuss this issue in the future.

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## Appendix

Given the BPS equation

$$\frac{1}{2}\varepsilon_{ijk}F_{jk} + D_i\Phi = 0 \tag{56}$$

the monopole solution, taking as zeroth order in  $\theta$  the Prasad-Sommerfield solution (21), is, up to order  $\theta^2$ ,

$$\begin{aligned}
A_i^{a(1)} &= 0 \\
\Phi^{a(1)} &= 0 \\
A_i^{0(1)} &= \theta_{ij} x_j \frac{1}{4r^2} W(W + 2F) \\
\Phi^{0(1)} &= 0 \\
A_i^{a(2)} &= \frac{1}{r^5} \left[ a_1(r) \theta^2 \epsilon_{aij} \hat{x}_j + a_2(r) (\theta \hat{x}) \epsilon_{aij} \theta_j + a_3(r) \epsilon_{ajk} \theta_i \theta_j \hat{x}_k \right. \\
&\quad \left. + a_4(r) (\theta \hat{x})^2 \epsilon_{aij} \hat{x}_j + a_5(r) (\theta \hat{x}) \epsilon_{ajk} \hat{x}_i \theta_j \hat{x}_k \right] \\
\Phi^{a(2)} &= \frac{1}{r^5} \left[ \phi_1(r) (\theta \hat{x}) \theta_a + \phi_2(r) \theta^2 \hat{x}_a + \phi_3(r) (\theta \hat{x})^2 \hat{x}_a \right] \\
A_i^{0(2)} &= 0 \\
\Phi^{0(2)} &= 0
\end{aligned} \tag{57}$$

Here

$$\begin{aligned}
\phi_1(r) &= -\frac{1}{4} rF + \frac{1}{4} r^2 F^2 - \frac{1}{8} r^3 F^3 + \frac{1}{4} rF(1 - rW)^2 \\
\phi_2(r) &= \frac{1}{8} - \frac{3}{8} rF + \frac{1}{8} r^2 F^2 - \frac{1}{4} (1 - rW)^2 + \frac{3}{8} rF(1 - rW)^2 + \frac{1}{8} (1 - rW)^4 \\
\phi_3(r) &= -\frac{1}{8} + \frac{7}{8} rF - \frac{5}{8} r^2 F^2 + \frac{1}{4} (1 - rW)^2 + \frac{1}{8} r^3 F^3 - \frac{7}{8} rF(1 - rW)^2 \\
&\quad - \frac{1}{8} rF^2(1 - rW)^2 - \frac{1}{8} (1 - rW)^4
\end{aligned} \tag{58}$$

and

$$\begin{aligned}
a_1(r) &= -\frac{1}{8} + \frac{1}{2} rF - \frac{1}{8} (1 - rW) - \frac{1}{2} rF(1 - rW) - \frac{1}{4} r^2 F^2(1 - rW) \\
&\quad + \frac{5}{8} (1 - rW)^2 + \frac{1}{4} rF(1 - rW)^2 - \frac{3}{8} (1 - rW)^3 - \frac{1}{4} rF(1 - rF)^3 \\
a_2(r) &= \frac{1}{4} + \frac{1}{2} rF - \frac{3}{8} r^2 F^2 - \frac{3}{4} (1 - rW) - \frac{1}{2} rF(1 - rW) \\
&\quad + \frac{1}{8} r^2 F^2(1 - rW) + \frac{3}{4} (1 - rW)^2 - \frac{1}{4} (1 - rW)^3
\end{aligned} \tag{59}$$

$$\begin{aligned}
a_3(r) &= -\frac{1}{8} - \frac{1}{4}rF - \frac{1}{8}r^2F^2 + \frac{3}{8}(1-rW) + \frac{1}{2}rF(1-rW) \\
&\quad + \frac{1}{8}r^2F^2(1-rW) - \frac{3}{8}(1-rW)^2 - \frac{1}{4}rF(1-rW)^2 \\
&\quad + \frac{1}{8}(1-rW)^3 \\
a_4(r) &= -\frac{1}{4} - \frac{3}{2}rF + \frac{1}{2} + \frac{1}{2}r^2F^2 + \frac{5}{4}(1-rW) + \frac{3}{2}rF(1-rW) \\
&\quad + \frac{1}{4}r^2F^2(1-rW) - \frac{7}{4}r^2F^2 - \frac{1}{4}rF(1-rW)^2 \\
&\quad + \frac{3}{4}(1-rW)^3 + \frac{1}{4}rF(1-rW)^3 \\
a_5(r) &= 0
\end{aligned} \tag{60}$$

with

$$\begin{aligned}
F(r) &= v_0 \coth(v_0 er) - \frac{1}{er} \\
W(r) &= \frac{1}{er} - \frac{v_0}{\sinh(v_0 er)}
\end{aligned} \tag{61}$$

and

$$\begin{aligned}
\theta_i &\equiv (1/2)\epsilon_{ijk}\theta^{jk} \\
\theta^2 &\equiv \theta_i\theta_i \\
(\theta\hat{x}) &\equiv \theta_i\hat{x}_i
\end{aligned} \tag{62}$$

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